Understanding Linear Algebra Geometrically

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What is a vector

To have anything at all you must have space to put things in.

We are tasked with storing random numerical information.

In any discipline where we work with numbers, there are many ways to store and represent information.

If I worked in construction I might want to represent palettes of bricks as tallys

Let's say I had 360 bricks stacked in palettes of 120



It may or may not be convenient to store the information above as a tally of 3: 

This fully represents our data as the only variable here is how many palettes we have; all the palettes will always have 120 bricks.

The way we represent information is trivial… unless we want to manipulate numbers

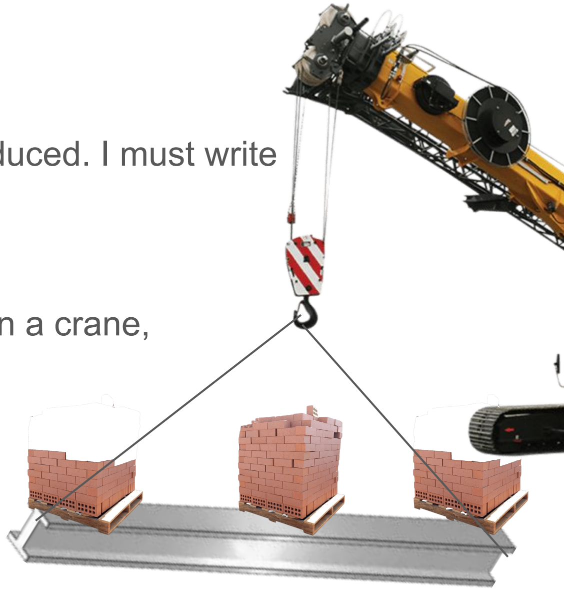
Let's say I am in charge of all the bricks at a construction site. I ordered 3 palettes

Unfortunately, the brickmakers stacked our palettes unevenly

one palette has 50 bricks, another has 110, another has 40.

No longer can I represent these with tallies, as a new dimension of data has been introduced. I must write them in a list:

50, 110, 40



I need to use 50% of our bricks for the project, but the palettes are balanced perfectly on a crane, they must be unloaded evenly, or else they'll all fall off and hit the workers.
I must find out how many to unload from each.
This is a simple undertaking for me as I'll just multiply all the numbers by 0.5

(50, 110, 40) * 0.5 -> 25, 55, 20

Problem solved. It's so easy to scale them.

This new project requires bricks blue and red. I asked for 270 red and 90 blue ones.

Unfortunately, the brickmakers once again maliciously stacked our palettes without order

This palette has 100 red bricks and 20 blue ones

This palette has 80 red bricks and 40 blue ones

This palette has 90 red bricks and 30 blue ones

Well, that's not bad, I have only one new variable in my data to consider.
Let's put two numbers for each palette representing colors like this

red 100 80 90
blue 20 , 40, 30

To remind myself im looking at a single order of bricks, I'll take out my draftsman pen and write it like this

$$\begin{bmatrix} \text{red} \\ \text{blue} \end{bmatrix} = \begin{bmatrix} 100 \\ 20 \end{bmatrix}, \begin{bmatrix} 80 \\ 40 \end{bmatrix}, \begin{bmatrix} 90 \\ 30 \end{bmatrix}$$

Feeling triumphant, I name this innovation after my first and last name, VecTor

Vector combinations

Just as I finished writing in my notebook, my boss calls me over and says:

"Hey Vec, we need exactly 220 red bricks and 80 blue bricks for this wall. Take from any wooden palettes we have, but don't waste any wood."

As he left, he took a big swig from his canteen and got liquid all over my notebook. I looked down at my notes

$$\begin{bmatrix} 100 & 80 & 90 \\ 20 & 40 & 30 \end{bmatrix}$$

My new goal was $\begin{bmatrix} 220 \\ 80 \end{bmatrix}$, 200 red bricks and 80 blue

As I stared at the smudged numbers, my eyes focused on a symbol on the canteen my boss had set down. Suddenly, everything around me disappeared and I was in a world of math.

$$\left[\begin{array}{ccc|c} 100 & 80 & 90 & 220 \\ 20 & 40 & 30 & 80 \end{array}\right]$$

I wrote what i was looking for on the right of my matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The canteen had this symbol on it, it's called the identity matrix, I must try to recreate it

$$\left[\begin{array}{ccc|c} 10 & 8 & 9 & 22 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

I multiplied row 1 (R1) by 1/10 and R2 by 1/10
To make the numbers easier

I **eliminated** the first entry by subtracting 5 times the second row from the first but wasn't this supposed to be a 1?

$$\left[\begin{array}{ccc|c} 0 & -12 & -6 & -18 \\ 2 & 4 & 3 & 8 \end{array}\right]$$

R1-5R2

I got back on track by **swapping** the first and second row

$$\left[\begin{array}{ccc|c} 2 & 4 & 3 & 8 \\ 0 & -12 & -6 & -18 \end{array}\right]$$

SWAP R1 R2

I **scaled** the first row by 1/2

$$\left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 4 \\ 0 & -12 & -6 & -18 \end{array}\right]$$

½ R1

I scaled the second row by doing ½ R2 to allow me to eliminate the 2 in the first row, getting me closer to the identity matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & \frac{3}{2} & 4 \\ 0 & 2 & 1 & 3 \end{array}\right]$$

½ R2

I successfully made a diagonal of all 1's by doing R1-R2 and scaling R2 by ½.
This is called **reduced row echelon form.**

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{2} & 1 \\ 0 & 1 & \frac{1}{2} & \frac{3}{2} \end{array}\right]$$

Something looks wrong, I made my identity matrix, but I am left over with a 3rd column. This is the **free variable.**

Looking back at our original problem, we were trying to find out how we could scale my set of palettes, (vectors,) to determine how much red and blue bricks I should take from each. Since we are talking about scale, we should assign a variable to each vector

$$a\begin{bmatrix}100\\20\end{bmatrix}+b\begin{bmatrix}80\\40\end{bmatrix}+c\begin{bmatrix}90\\30\end{bmatrix}=\begin{bmatrix}220\\80\end{bmatrix}$$

We know that our solution should have 3 variables, each variable scales one set of bricks. As we know from before, it's cleaner to write a bunch of variables as a vector. Let's do that shall we?

$$\begin{bmatrix}100 & 80 & 90\\20 & 40 & 30\end{bmatrix}\begin{bmatrix}\mathrm{a}\\\mathrm{b}\\\mathrm{c}\end{bmatrix}=\begin{bmatrix}220\\80\end{bmatrix}$$

Wow, it's almost as if we can multiply a vector onto a **matrix**. A **matrix** is a collection of vectors, it's what the boss accidentally made. If you want to see matrix multiplication in action, go to matrixmultiplication.xyz

Since we now know we can multiply vectors and matrices, we can set up our solution like this

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} \\ 0 & 1 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix}$$

This is simply representing a system of equations. Now that you know how the variables correspond with the columns from matrix multiplication, let's set it up as an actual system of equations

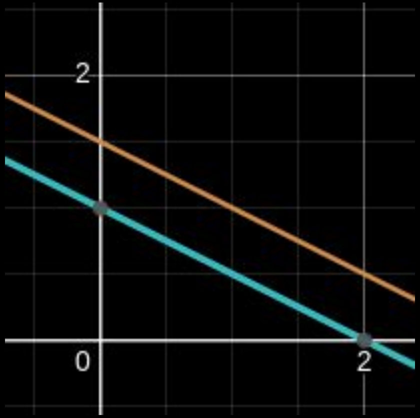
$$a + 0 + \frac{1}{2}c = 1$$

$$0 + b + \frac{1}{2}c = \frac{3}{2}$$

Notice how we can solve for a and b in terms of c. $a = 1 - \frac{1}{2}c, \qquad b = \frac{3}{2} - \frac{1}{2}c$

Here’s where it gets helpful. We already determined C was a free variable since it was off doing it’s own thing and wasn’t with the rest of the identity matrix. This means we have the freedom to set it to whatever we want to determine the values of a and b.

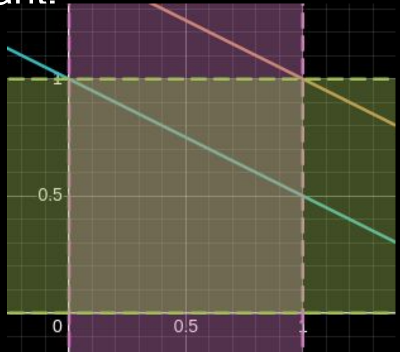
The boss asked us to find any amount of bricks as long as it had 220 red and 80 blue, and as long as we don't leave any empty palettes as litter. There are clearly hundreds of ways you could fill these requirements, so one of the best way to represent multiple solutions is through a graph. Let's graph $a = 1 - \frac{1}{2}c$ and $b = \frac{3}{2} - \frac{1}{2}c$ in 2d



Hmm, the lines don't intersect. Does this mean there really is no solution to the problem?!

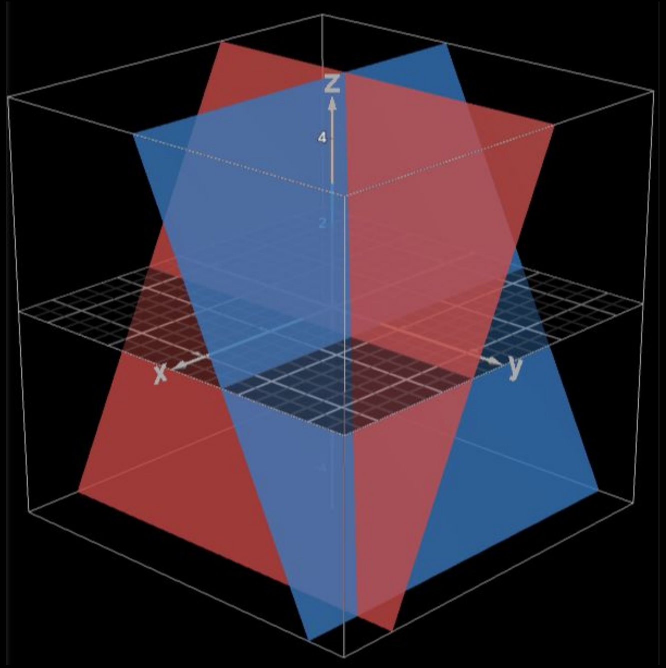
When interpreting data, you must step back and consider the solution.
In this case, each x value of c provides 2 solutions, one for a and one for b. This is exactly what we want.

To be more accurate, we should limit our data to c > 0 and y > 0.



Bonus: 3d representation

$$a = 1 - 0.5c, \quad b = 1.5 - 0.5c, \quad c = c$$

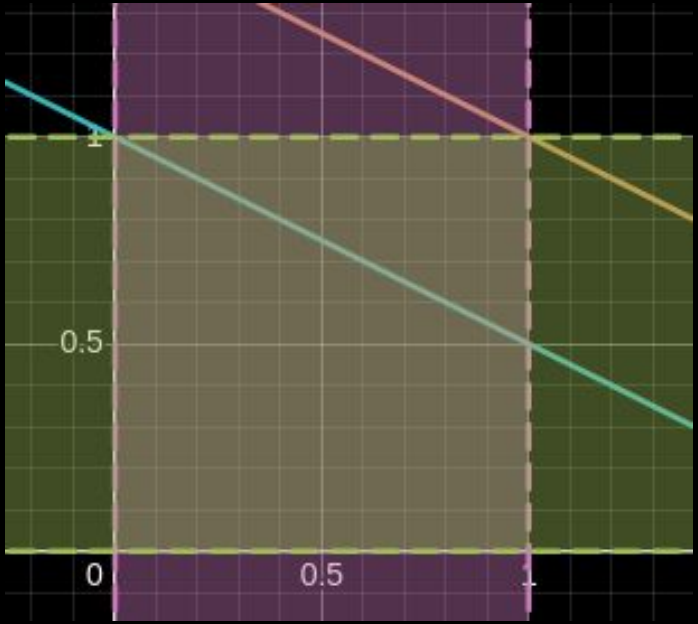


All places the planes intersect are solutions, just like on the 2d graph. A is red and b is blue.

Did you notice something?

In our inequality with the graph, there was no solution for a and b ,where a, b, and c were greater than 0 (boss requirement) and less than 1 (you can't physically have more bricks than what's on the palette).

Purple is $0 < x < 1$
Yellow is $0 < y < 1$ (y is a and b)



Look at how the red line is not within our square of solutions, **there is no x value, aka c value, that would allow us to take enough bricks from all palettes without leaving an empty palette.**

You break free from your math induced coma in a sweat

“I cannot solve this, boss!” you cry,

Gathered around you is the entire construction crew, along with your boss.

Before you could protest further, you are sent home for the day to recover.

I went home that night, mind racing.
In my garage,
I built the scaling machine.
It allows me to change the scale of
any vectors I run into in real life

At work, I am asked to oversee 3 workers who can rearrange bricks in different patterns.

Worker A always moves 2 bricks from the first stack and 1 from the second

Worker B moves 4 bricks from the first and 2 from the second

Worker C moves 1 brick from the first and 3 from the second

Each worker's 'movement' can be represented by vectors

$$\mathbf{A} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Let's put that in a matrix

$$\begin{bmatrix} 2 & 4 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

All that running back and forth seems like such a waste, surely there's a way to find a redundant worker and replace them?

We know from before that the identity matrix is a diagonal of zeros, essential a square. A square cannot fit perfectly within a rectangle whose width is greater than its height. For this reason, we found that we were left with a free variable column/palette earlier. That matrix was a 2x3 matrix, 2 rows and 3 columns. So is our current matrix. This must mean one of the columns / workers in our case is free as well. Let's try to figure it out.

These guys can fit an identity matrix

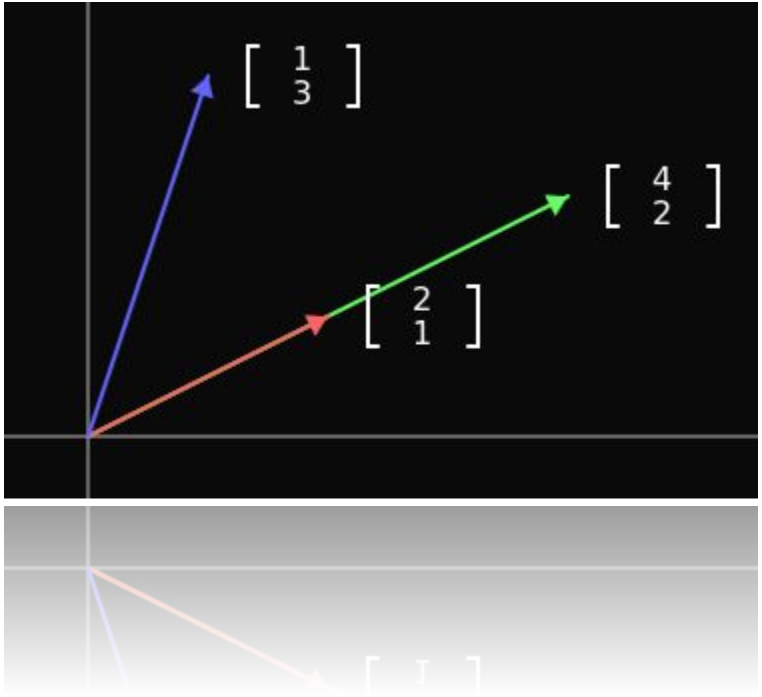
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$$

It is IMPOSSIBLE to include this column / variable in the identity

My idea is to remove the free worker and use my scaler to scale up the other workers to achieve the same work. The problem is, I’m not sure how to visualize vectors in relation to one another.

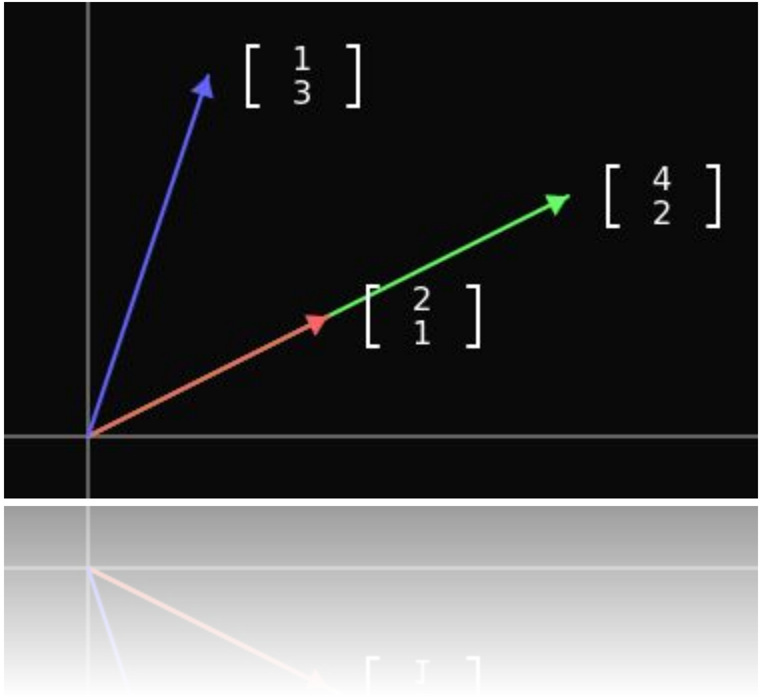
Fortunately, there is a solution. There’s something called Vector space. Let's take a look at it



Wow, this is an entirely new way to interpret vectors in relation to each other.
Can you guess which one is the free variable?

Here’s a new term, if you were to guess which two vectors were independent of each other and which were dependent, which would you guess?

The green and red vectors are dependent as they can be written as scaled versions of each other.



The blue and green, and blue and red vectors are linearly independent, they cannot be scaled to become identical to each other.

I could fire worker A and scale worker B whenever needed. That gives us this new matrix:

$$\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Since this matrix only contains the crucial workers, it means that this matrix contains the **basis vectors**

Basis vectors are the essential vectors in a matrix that perform the same utility as the original matrix without excess vectors, like free variables. Notice how the free variable was eliminated? Let's do the reduction we did earlier on this matrix:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The two vectors we have here are pivot columns, this means that they are elements of the identity matrix.

Let's now see what reducing the original matrix looks like:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Notice how if we switch the columns around we can get that identity matrix, meaning B could also be a free variable. The rule of thumb is any col that doesn’t have a leading number is a free variable. However, we can remove either col A or B to get an identity matrix.
We can either fire worker A and scale B’s work by 1.5 to accommodate for A’s work, or fire worker B and scale A’s work by 3

Basis vectors

Basis vectors are important to understanding vectors space. The vector space of a matrix or a set of vectors is every single place you can reach by scaling the vectors and combining them. Let's say we had a matrix with 2 basis vectors and 3 free variables:

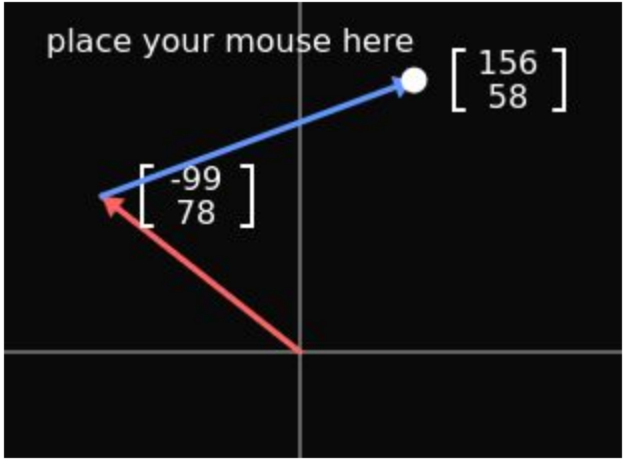
$$\begin{bmatrix} * & * & 0 & 0 & 0 \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Columns 1 and 3 are pivot variables. The points that can be formed by a combination of all the vectors in this matrix are the same exact points that can be reached with a matrix only containing these pivot variables, aka basis vectors.

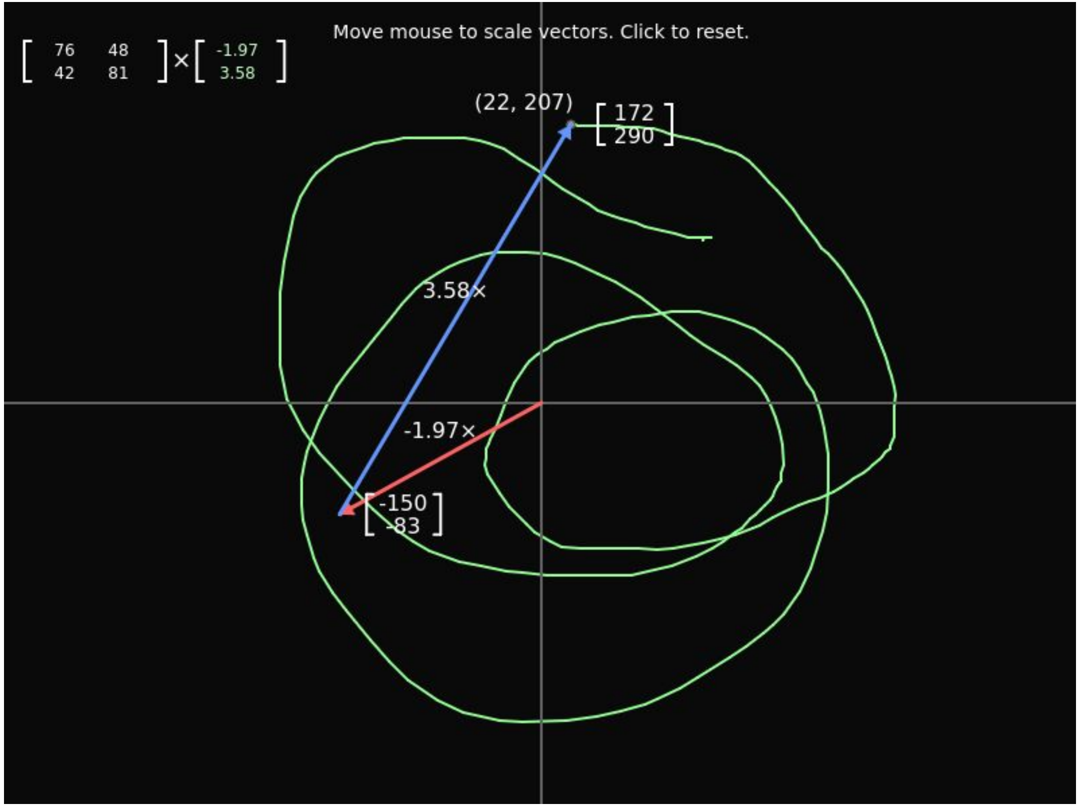
$$\begin{bmatrix} * & 0 \\ 0 & * \\ 0 & 0 \end{bmatrix}$$

The space that can be reached by all the columns is called the col space.

Try experimenting with the concept of vector space and creating a col space with basis vectors on my site

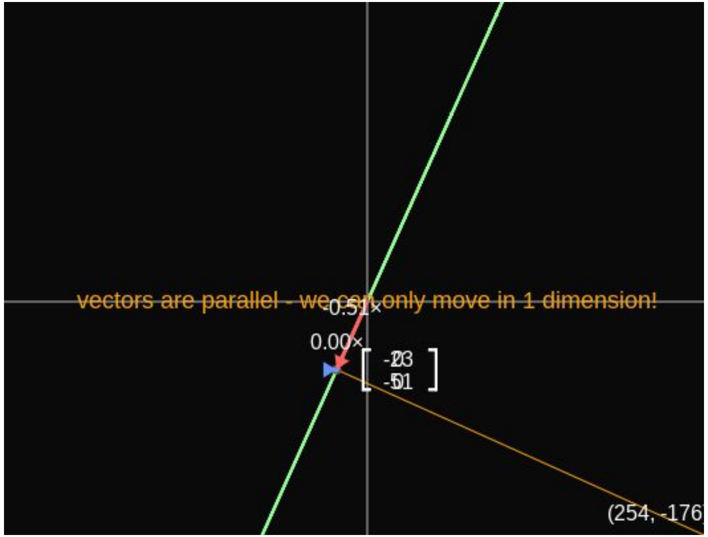


https://luroxin.com/linearalgebra/vectorspace/



https://luroxin.com/linearalgebra/vectorarea/

You may or may not have noticed that in the simulation, making 2 parallel basis vectors results in you not being able to hit every single point on your screen. If you want to try this, go back to either simulator and try making 2 identical vectors, or at least make them parallel.



When you create two non parallel vectors, you can move in 2D (2 dimensions) perfectly. We call this the Rank, or col space dimension. Notice how despite having 2 vectors, the rank can be less than 2. In this case, we can only move in 1d, so the rank is 1.

Linear Transformation

You may have noticed in the demo we expressed the scale as a vector (in green)

$$\begin{bmatrix} 63 & -13 \\ 31 & 38 \end{bmatrix} \times \begin{bmatrix} 2.22 \\ 3.61 \end{bmatrix}$$

This is a form of linear transformation. If you put in a point/vector into the linear transformation. The way I like to think about it is as a function. It displaces your points/vectors systematically

$$f(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} 2.22 \\ 3.61 \end{bmatrix}$$

In that example, we saw two vectors being transformed by another vector:

$$\begin{bmatrix} 2.22 \\ 3.61 \end{bmatrix} * \begin{bmatrix} 63 \\ 31 \end{bmatrix} \text{ Or } \begin{bmatrix} 2.22 & 3.61 \end{bmatrix} \begin{bmatrix} 63 \\ 31 \end{bmatrix}$$

$$\begin{bmatrix} 2.22 & 3.61 \end{bmatrix} \begin{bmatrix} -13 \\ 38 \end{bmatrix}$$

We can transform vectors by matrices as well:

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -13 \\ 38 \end{bmatrix}$$

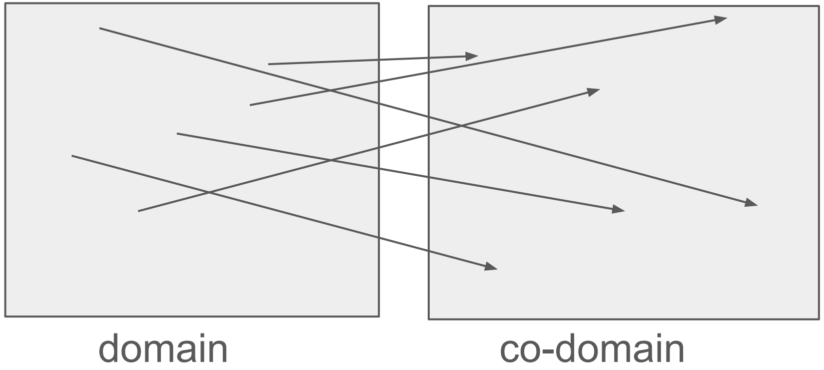
Try putting this in [matrixmultiplication.xyz]() to visually understand how to multiply these

This matrix would double the vector. This is called a **transformation matrix**

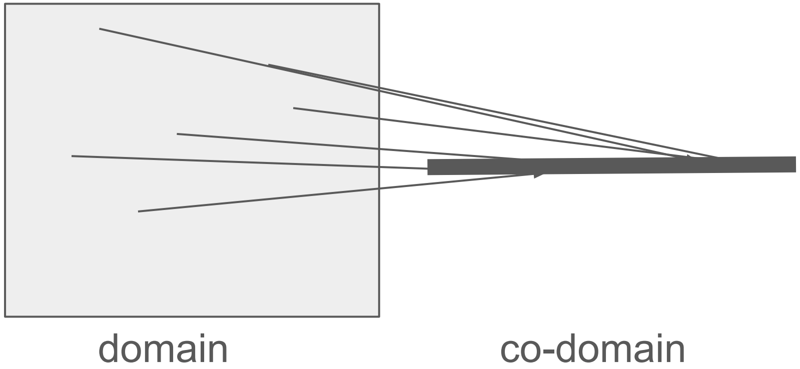
Null Space

When we transform a set of vectors, whether using a single vector or a matrix, we could be changing the space in some way. For example, let's say we had vectors with rank 2. If we transformed them in a way for them to become parallel, it would mean we are now in rank 1. We took a 2d space and turned it into a 1d space. These two different spaces are completely different. We call the original space the **domain** and the resulting space the **codomain**.

Non destructive transformation, rank stays the same
Each point in the domain can have a unique* point
in the co-domain



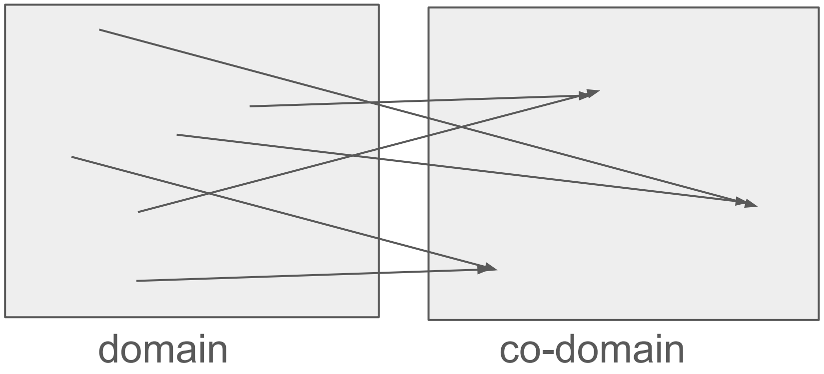
destructive transformation, rank changes
Each point in the domain cannot have unique point
in the co-domain, there must be overlap



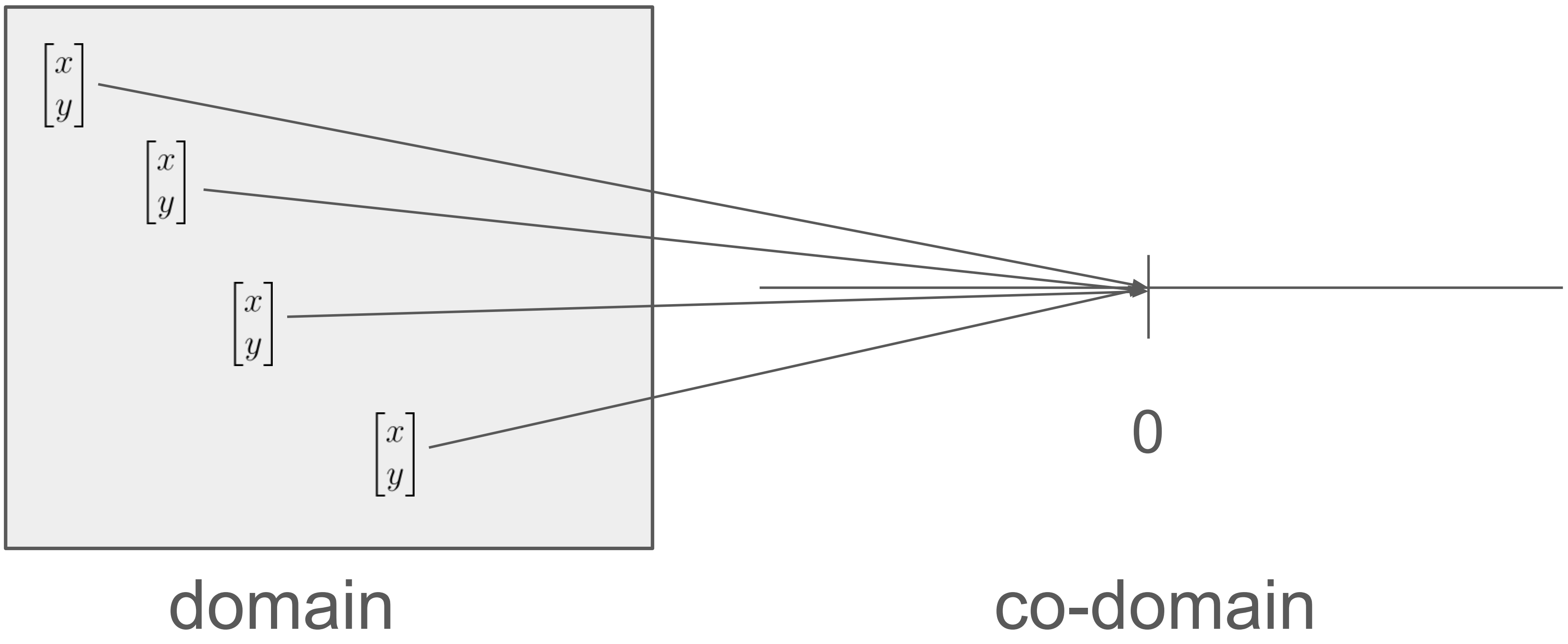
*$ see speaker notes

Important note

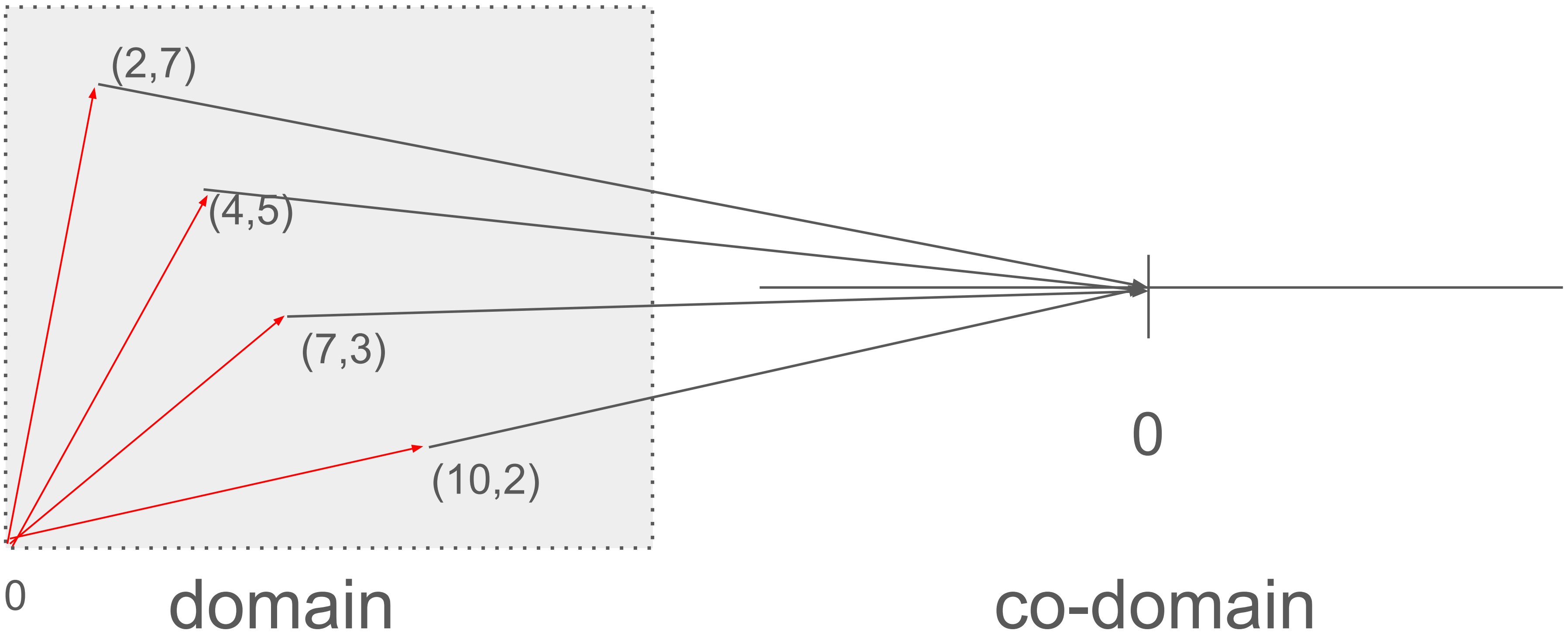
Just because a transformation keeps the same rank, does not mean that each input has a unique output.



Nullspace is the vector space, or simply set of vectors, in the domain that end up at 0 in the codomain



remember that these are vectors, or points

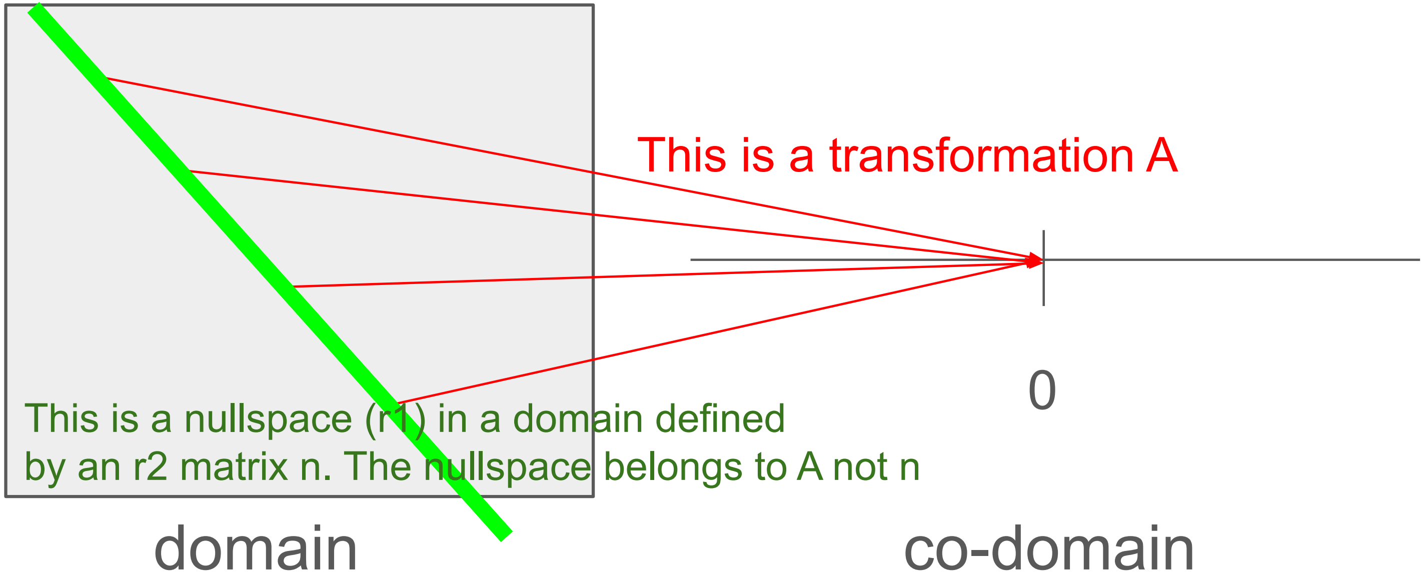


Recall a transformation can be a matrix. If we were to think about it as a function, we could write it like this:

$$f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \end{bmatrix}$$

Where A is a transformation matrix, transforming an input vector. Textbooks call this the matrix equation, written like this: $Ax = b$ where A is a transformation matrix, x is a vector, and b is what the vector is after it gets transformed by A

Simply based on a given transformation (matrix), you can figure out which input vectors will go to zero when transformed. This set of input vectors is a space.



Important distinction/misconception

A vector space can be defined by a matrix.

A transformation can be defined by a matrix.

If you are asked to find basis vectors for the nul space of a matrix, it's treating the matrix like a transformation matrix.

A vector space has no null space as it does not have the capacity to map vectors to anything.

A nullspace of r1 is in a domain defined by an r2 matrix n. The codomain is created by transforming the col space of n by A. The nullspace belongs to A not n (the ranks are just for example, this is true for higher ranks)

I wanted to explain image deblurring and nullspace, but I did not have time. I will still do it

- [x] standard representing a point from 2 non unit vectors via scaling
- [x] represent that you can touch all places using the above technique
- [ ] how row reduction changes the col space but not the row space/dimention/rank/span (this is really important because i dont see anyone else graphically displaying it)
- [ ] matrix of a one to one linear transformation. I really like what he did with his nonlinear matrix transformations, i can do something like that
- [ ] represent elementary matricies geometrically
- [x] I know blue brown did this already, but represent the null space. im not sure if he actually shows going from the domain to the codomain, i will be doing that though
- [ ] show that matricies that are independent do not flatten, and matracies that are dependent, the vectors that are free variables flatten down and the basis vectors really define the space. genuinely no clue how to do this in 2d I will need to do this in 3d
- [ ] I will have to do malo's favorite thing, which is the economics supply and demand thing. I can do a simulation. what confused me the most is how an industry can take up its own resources. i can animate that
- [ ] subspace, perhaps I can show how the nullspace is a subspace of A
- [ ] obviously show image and geometry and stuff being transformed by matrix, should be super easy. go a step beyond what nicky cage did and actually describe each parameter. I had to experiment a ton to figure it out
- [ ] matrix factorization, I have no idea how to represent this graphically. go look on page 126, i understand that the three matrix equations Ly=b and Ux=y and Ax=B are traverseable using multiplication, but i dont know how to intuitively represent this interactively
- [ ] I'm not sure if this is possible, but yk how that one website did an animation of matrix multiplication and showed the result in real time? maybe I can show two matricies or images graphically and show their matricies being multiplied in real time and the result. also something I disliked about that demo was it was hard to follow because they multiplied the entire matrix at once, while I usually do row1xcol1, row1xcol2, row2xcol1, row2xcol2. that is probably sufficiant to be unique, no need to go crazy on each of these.
- [ ] not sure how to do this but somehow show that a vector can be within the span of a matrix. this is when you augment the matrix with the vector you are checking and you check if its consistant or not
- [ ] try to do something with consistant, unique and inconsistant?
- [ ] 100% show graphically what different determinants mean. that interactive linear textbook did that already but I can show it better
- [ ] smth to do with rank idk
- [ ] I wanna do the dimention flattening demo, flatland, and tie it into what happens when you flatten a matrix or smth

Section that has brick piles of 3 different colors: r,g, and b. The character will use this to explore image kernels. After that, to do more advance image kernels, the palettes will have variables beyond color, making each pixel/pallette be 4+x1 size. I will then directly show this behavior on a real image

Vector and Raster Graphics